The Set of Zero Divisors of a Factor Ring

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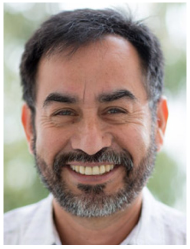

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Abstract

We show that if A is a commutative ring with unity and $\mathfrak{a}$ is an ideal of A which is a finite product of relative prime ideals $\mathfrak{b}_i$ then the factor ring $A/\mathfrak{a}$ is a direct sum of ideals $\mathfrak{a}_i/\mathfrak{a}$. Moreover, each ideal $\mathfrak{a}_i/\mathfrak{a}$ endowed with addition and multiplication modulo $\mathfrak{a}$ is a ring isomorphic to the factor ring $A/\mathfrak{b}_i$. We give examples when A is the ring of integers $\mathbb{Z}$, the Gaussian integers $\mathbb{Z}[\mathbf{i}]$, or a ring of polynomials $\mathbb{F}_q[x]$ over a finite field with q elements $\mathbb{F}_q$.

1 Introduction

Throughout this paper A will be a commutative ring with a non-zero multiplicative identity. The group of units of A will be denoted by $U(A)$ and the set of zero divisors together with the zero element will be denoted by $Z(A)$. If A is finite then the set $Z(A)$ is the complement of $U(A)$, Lemma 1. In [2] and [6], the authors show that if $A = \mathbb{Z}_a$, the ring of integers modulo a, then there exist positive integers n such that $U(\mathbb{Z}_a)$ can be mapped isomorphically to a group D contained in the set $Z(\mathbb{Z}_n)$ with the group structure of D given by multiplication modulo n. Here we will show, by means of the Chinese Remainder Theorem, that there exist positive integers n such that the ring $\mathbb{Z}_a$ can be mapped, ring isomorphically, to a ring R contained in the set $Z(\mathbb{Z}_n)$ with the ring structure of R given by addition and multiplication modulo n. This shows that the group $U(\mathbb{Z}_a)$ is isomorphic to $U(R) \subset R \subset Z(\mathbb{Z}_n)$.

2 Elements of Ring Theory

An element x of a A is *nilpotent* if and only if there exists a positive integer n such that $x^n = 0$. An element e of a A is *idempotent* if and only if $e^2 = e$. If e_1 and e_2 are nonzero idempotent elements of A, we say that they are *orthogonal* if and only if $e_1 \cdot e_2 = 0$. We remark that if an element y of A is both nilpotent and idempotent then $y = 0$. We say that two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A are *relatively prime* or *coprime* if and only if $\mathfrak{a} + \mathfrak{b} = A$. A proper ideal $\mathfrak{m}$ of A is a *maximal* ideal if and only if for any ideal $\mathfrak{a}$ of A such that $\mathfrak{m} \subseteq \mathfrak{a} \subseteq A$ either $\mathfrak{m} = \mathfrak{a}$ or $\mathfrak{a} = A$. A ring A is called a *local ring* if and only if it has only one proper maximal ideal $\mathfrak{m}$. If $A_1, A_2, \ldots, A_k$ are rings then the *direct product* of these rings is the ring

$$A_1 \times A_2 \times \cdots A_k$$

with component-wise addition and multiplication.

Lemma 1. *If A is a finite ring then $Z(A) = A \setminus U(A)$.*

Proof. First, we observe that $U(A) \cap Z(A) = \emptyset$ since a unit may not be a zero divisor. Next, let $a \neq 0$ be an element of A and define the homomorphism $\mu_a : A \to A$ (A consider as a commutative group under addition) by $\mu_a(x) = ax$. If μ_a is not injective then μ_a has a nonzero kernel so there exists $b \neq 0$ in A such that $\mu_a(b) = ab = 0$ and it follows that a is a zero divisor.
Otherwise, since A is finite, if μ_a is injective μ_a must be onto. Therefore, there exists $c \neq 0$ in A such that $\mu_a(c) = ac = 1$. This implies that a is a unit. $\square$

Lemma 2. *If A is a local ring then the only idempotent elements of A are either* 0 *or* 1.

Proof. Let $\mathfrak{m}$ be the maximal ideal of A and e an idempotent of A. We observe that in a local ring an element x is either in $\mathfrak{m}$ or is a unit. Also, it is not possible to have both e and $1 - e$ be elements of $\mathfrak{m}$ since this implies that $1 = e + (1 - e)$ is in $\mathfrak{m}$ which contradicts the fact that $\mathfrak{m}$ is a proper ideal. Therefore, either e is a unit or $1 - e$ is a unit. Since $e = e^2$, we have $e \cdot (1 - e) = 0$. If e is a unit then $1 - e = 0$, this implies that $e = 1$. On the other hand, if $1 - e$ is a unit then $e = 0$. $\square$

Lemma 3. *Let $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_k$, be ideals of a ring A. Assume that $\mathfrak{b}_i + \mathfrak{b}_j = A$ whenever $i \neq j$. Let $\mathfrak{a}_i = \prod_{j \neq i} \mathfrak{b}_j$. Then, for all $i = 1, 2, \ldots, k$, $\mathfrak{b}_i + \mathfrak{a}_i = A$.*

Proof. For every $j \neq i$ let $a_j \in \mathfrak{b}_j$ and $b_i \in \mathfrak{b}_i$ be such that $a_j + b_i = 1$. Then,

$$a_i = \prod_{j \neq i} a_j = \prod_{j \neq i} (1 - b_i) \equiv 1 \bmod \mathfrak{b}_i.$$

This implies that $a_i - 1 \in \mathfrak{b}_i$. So, there exists $b_i \in \mathfrak{b}_i$ such that $a_i + b_i = 1$. This shows that $\mathfrak{a}_i + \mathfrak{b}_i = A$. $\square$

Lemma 4. *Let $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_k$ be ideals of ring A. Assume that $\mathfrak{b}_i + \mathfrak{b}_j = A$ whenever $i \neq j$. Then,*

$$\prod_{i=1}^{k} \mathfrak{b}_i = \bigcap_{i=1}^{k} \mathfrak{b}_i.$$

Proof. We will use induction on k. If $k = 2$ then $\mathfrak{b}_1$ is relatively prime to $\mathfrak{b}_2$ by assumption. Let $b_1 \in \mathfrak{b}_1$ and $b_2 \in \mathfrak{b}_2$ be such that $b_1 + b_2 = 1$. If $x \in \mathfrak{b}_1 \cap \mathfrak{b}_2$ then $x = xb_1 + xb_2 \in \mathfrak{b}_1\mathfrak{b}_2$. Since $\mathfrak{b}_1\mathfrak{b}_2 \subseteq \mathfrak{b}_1 \cap \mathfrak{b}_2$, it follows that $\mathfrak{b}_1\mathfrak{b}_2 = \mathfrak{b}_1 \cap \mathfrak{b}_2$. Assume that $k \geq 3$. By Lemma 3, $\mathfrak{b}_1$ is relatively prime to $\prod_{i \geq 2} \mathfrak{b}_i$. By the case $k = 2$

$$\mathfrak{b}_1 \cdot \prod_{i \geq 2}^{k} \mathfrak{b}_i = \mathfrak{b}_1 \bigcap \prod_{i \geq 2}^{k} \mathfrak{b}_i.$$

By induction hypothesis

$$\prod_{i \geq 2}^{k} \mathfrak{b}_i = \bigcap_{i \geq 2}^{k} \mathfrak{b}_i.$$

The last two equations show that

$$\prod_{i=1}^{k} \mathfrak{b}_i = \bigcap_{i=1}^{k} \mathfrak{b}_i.$$

This completes the proof. $\square$

Lemma 5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ ideals of A such that $\mathfrak{a}+\mathfrak{b}=A$. Let $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ be such that $a+b=1$. Let $A/\mathfrak{a}$, $A/\mathfrak{b}$, $A/\mathfrak{ab}$ be the factor rings of A by the ideals $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{ab}$ respectively. Then, the ring morphism*

$$\pi : A/\mathfrak{ab} \to A/\mathfrak{a} \times A/\mathfrak{b},\ (x \bmod \mathfrak{ab}) \mapsto (x \bmod \mathfrak{a}, x \bmod \mathfrak{b})$$

is an isomorphism and the ideals $\mathfrak{a}/\mathfrak{ab}$ and $\mathfrak{b}/\mathfrak{ab}$ endowed with addition and multiplication modulo $\mathfrak{ab}$ are rings with unity. Moreover, the unity of $\mathfrak{a}/\mathfrak{ab}$ is $a \bmod \mathfrak{ab}$, the unity of $\mathfrak{b}/\mathfrak{ab}$ is $b \bmod \mathfrak{ab}$ and the ring morphisms

$$\begin{aligned}
&\mathfrak{a}/\mathfrak{ab} \to A/\mathfrak{b},\ (x \bmod \mathfrak{ab}) \mapsto (x \bmod \mathfrak{b}) \quad \textit{and} \\
&\mathfrak{b}/\mathfrak{ab} \to A/\mathfrak{a},\ (y \bmod \mathfrak{ab}) \mapsto (y \bmod \mathfrak{a})
\end{aligned}$$

are isomorphisms.

Proof. The ring morphism

$$A \to A/\mathfrak{a} \times A/\mathfrak{b},\ (x \mapsto (x \bmod \mathfrak{a}, x \bmod \mathfrak{b})$$

is well defined and surjective and its kernel is $\mathfrak{a} \cap \mathfrak{b}$. By Lemma 3, $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$. This implies that π is an isomorphism. Since $a+b=1$ we have

$$(b \bmod \mathfrak{ab}) \mapsto (b \bmod \mathfrak{a}) = (1 - a \bmod \mathfrak{a}) = (1 \bmod \mathfrak{a}).$$

Now we show that $\mathfrak{b}/\mathfrak{ab} \to A/\mathfrak{a}$ is surjective. Let $(x \bmod \mathfrak{a}) \in A/\mathfrak{a}$. We have $bx \in \mathfrak{b}$ and

$$\begin{aligned}
(bx \bmod \mathfrak{ab}) \mapsto (bx \bmod \mathfrak{a}) &= (b \bmod \mathfrak{a})(x \bmod \mathfrak{a}) \\
&= (1 \bmod \mathfrak{a})(x \bmod \mathfrak{a}) \\
&= (x \bmod \mathfrak{a}).
\end{aligned}$$

Next we show that $\mathfrak{b}/\mathfrak{ab} \to A/\mathfrak{a}$ is injective. If y_1 and y_2 are in $\mathfrak{b}$ and

$$(y_1 \bmod \mathfrak{a}) = (y_2 \bmod \mathfrak{a})$$

then $y_1 - y_2 \in \mathfrak{a}$. It follows that $y_1 - y_2 \in \mathfrak{a} \cap \mathfrak{b} = \mathfrak{ab}$. This implies that

$$(y_1 \bmod \mathfrak{ab}) = (y_2 \bmod \mathfrak{ab}).$$

and it follows that the morphism is injective. Therefore, the morphism is an isomorphism. $\square$

Theorem 6 (Chinese Remainder Theorem)**.** *Let $\mathfrak{b}_1, \mathfrak{b}_2, \ldots \mathfrak{b}_k$, be ideals in A. Set $\mathfrak{a} = \prod \mathfrak{b}_i$ and assume that $\mathfrak{b}_i + \mathfrak{b}_j = A$ whenever $i \neq j$. Then, the ring morphism*

$$\pi : A/\mathfrak{a} \to A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \cdots \times A/\mathfrak{b}_k,\ \pi(x \bmod \mathfrak{a}) = (x \bmod \mathfrak{b}_1, x \bmod \mathfrak{b}_2, \ldots, x \bmod \mathfrak{b}_k)$$

is a ring isomorphism.

Proof. We will use induction on k. If $k = 1$ there is nothing to prove. Assume that $k = 2$. Since $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are relatively prime the result follows from Lemma 5. Suppose that $k \geq 3$. By Lemma 3, $\mathfrak{b}_1$ is relatively prime to the product $\prod_{i\geq 2}^{k} \mathfrak{b}_i$. Therefore,

$$\begin{aligned} A/\mathfrak{a} &= A/(\mathfrak{b}_1 \cdot (\mathfrak{b}_2 \cdots \mathfrak{b}_k)) \\ &\simeq A/\mathfrak{b}_1 \times A/(\mathfrak{b}_2 \cdots \mathfrak{b}_k), \text{ (by Lemma 5)} \\ &\simeq A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \mathfrak{b}_3 \times \cdots \times A/\mathfrak{b}_k \text{ (by induction hypothesis).} \end{aligned}$$

□

Corollary 7. *Let*

$$\pi : A/\mathfrak{a} \to A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \cdots \times A/\mathfrak{b}_k,\ \pi(x \bmod \mathfrak{a}) = (x \bmod \mathfrak{b}_1, x \bmod \mathfrak{b}_2, \ldots, x \bmod \mathfrak{b}_k)$$

be as in Theorem 6, $\pi_i : A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \cdots \times A/\mathfrak{b}_k \to A/\mathfrak{b}_i$ be the canonical projection, and $\mathfrak{a}_i = \prod_{j\neq i} \mathfrak{b}_j$. Then $\mathfrak{a}_i/\mathfrak{a}$ endowed with addition and multiplication modulo $\mathfrak{a}$ is a ring with unity and

$$\pi_i \circ \pi\ |_{\mathfrak{a}_i/\mathfrak{a}} \colon \mathfrak{a}_i/\mathfrak{a} \to A/\mathfrak{b}_i$$

is an isomorphism.

Proof. This follows from Lemma 5 since $\mathfrak{a}_i$ is relatively prime to $\mathfrak{b}_i$ and $\mathfrak{a} = \mathfrak{a}_i\mathfrak{b}_i$. □

Corollary 8. *Let $\mathfrak{a}_i$ be as in Corollary 7 and $e_i \in \mathfrak{a}_i$ be such that $e_i \equiv 1 \bmod \mathfrak{b}_i$. Then e_i is an idempotent in $\mathfrak{a}_i$ for all $i = 1, 2, \ldots, k$, e_i is orthogonal to e_j whenever $i \neq j$, and $e = e_1 + e_2 + \cdots + e_k \equiv 1 \bmod \mathfrak{a}$. Moreover, e_i is the multiplicative identity of $\mathfrak{a}_i/\mathfrak{a}$*

Proof. Since the map $\pi_i \circ \pi\ |_{\mathfrak{a}_i/\mathfrak{a}}$ is an isomorphism and

$$\pi_i \circ \pi\ |_{\mathfrak{a}_i/\mathfrak{a}} (e_i^2 \bmod \mathfrak{a}) \equiv 1 \bmod \mathfrak{a} \equiv \pi_i \circ \pi\ |_{\mathfrak{a}_i/\mathfrak{a}} (e_i \bmod \mathfrak{a})$$

we have $e_i^2 \equiv e_i \bmod \mathfrak{a}$ for all $i = 1, 2, \ldots, k$. Now, if $i \neq j$ then $e_i \cdot e_j \in \mathfrak{a}_i\mathfrak{a}_j \subseteq \mathfrak{a}$ and it follows that $e_i \cdot e_j \equiv 0 \bmod \mathfrak{a}$. Also, since

$$\pi(e \bmod \mathfrak{a}) = (1 \bmod \mathfrak{b}_1, 1 \bmod \mathfrak{b}_2, \ldots, 1 \bmod \mathfrak{b}_k) = \pi(1 \bmod \mathfrak{a})$$

we have $e \equiv 1 \bmod \mathfrak{a}$. That e_i is the multiplicative identity of $\mathfrak{a}_i/\mathfrak{a}$ follows from Lemma 5. □

Corollary 9. *Let $\mathfrak{a}_i$ and $e_i \in \mathfrak{a}_i$ be such that $e_i \equiv 1 \bmod \mathfrak{b}_i$, then $A/\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2 + \cdots + \mathfrak{a}_k$ and every element $x \in A$ can be uniquely written as $xe_1 + xe_2 + \cdots + xe_k \equiv x \bmod \mathfrak{a}$. That is, A is the direct sum of the ideals $\mathfrak{a}_i$.*

Proof. Since $e_1 + e_2 + \cdots + e_k \equiv 1 \bmod \mathfrak{a}$ it follows that $xe_1 + xe_2 + \cdots + xe_k \equiv x \bmod \mathfrak{a}$. To show that this representation is unique it suffices to show that if $xe_1 + xe_2 + \cdots + xe_k \equiv 0 \bmod \mathfrak{a}$ then $x \equiv 0 \bmod \mathfrak{a}$. Since

$$0 \bmod \mathfrak{a} = \pi_i \circ \pi\ |_{\mathfrak{b}_i/\mathfrak{a}} (0) = \pi_i \circ \pi\ |_{\mathfrak{b}_i/\mathfrak{a}} (xe_1 + xe_2 + \cdots + xe_k) = xe_i \bmod \mathfrak{a}_i = x \bmod \mathfrak{a}_i$$

it follows that $x \in \mathfrak{a}_i$ for all $i = 1, 2, \ldots, k$ so $x \in \cap_{i=1}^{k} \mathfrak{a}_i = \mathfrak{a}$. This shows that $x \equiv 0 \bmod \mathfrak{a}$. □

3 Factor rings of the ring of integers

Let $\mathbb{Z}_a$ be the ring of integers modulo a. The set $U(\mathbb{Z}_a)$ consists of all the elements in $\mathbb{Z}_n$ which are relatively prime to n. The number of elements of $U(\mathbb{Z}_a)$ is $\varphi(a)$, where $\varphi(a)$ is the Euler function. $U(\mathbb{Z}_a)$ is a group under multiplication modulo a and its group structure is well known. Recall that the set $Z(\mathbb{Z}_a)$ is the complement of $U(\mathbb{Z}_a)$ in $\mathbb{Z}_a$. The nonzero elements of $Z(\mathbb{Z}_a)$ are zero divisors under multiplication modulo a. The set $Z(\mathbb{Z}_a)$ is closed under multiplication modulo a, since the product of zero divisors is a zero or a zero divisor. At first sight, it seems that there is no hope of finding subsets of $Z(\mathbb{Z}_a)$ that could be groups under multiplication modulo a. However, as we will show in the example below, there are subsets of $Z(\mathbb{Z}_a)$ that are not only ideals of $\mathbb{Z}_a$ but are rings with unity when endowed with addition and multiplication modulo a and the units of these rings are groups under multiplication modulo a. See [2], [6]. The example below illustrates the central theme of this paper.

Example 10. *Consider the ring $\mathbb{Z}_{360}$. The sets*

$$\begin{aligned} Z_5 &= \{0, 72, 144, 216, 288\} = 72\mathbb{Z}_{360}, \\ Z_8 &= \{0, 45, 90, 135, 180, 225, 270, 315\} = 45\mathbb{Z}_{360}, \text{ and} \\ Z_9 &= \{0, 40, 80, 120, 160, 200, 240, 280, 320\} = 40\mathbb{Z}_{360} \end{aligned}$$

are principal ideals of $\mathbb{Z}_{360}$. These ideals are rings with unity under addition and multiplication modulo 360. The multiplicative identities of the ideals Z_5, Z_8 and Z_9 are $e_5 = 216 \bmod 360$, $e_8 = 225 \bmod 360$, and $e_9 = 280 \bmod 360$ respectively. The element 0 is the additive identity for them. The ring isomorphisms and their inverses are given below.

$$\begin{array}{lcllcl} \theta_{216} : Z_5 & \to & \mathbb{Z}_5 & \theta_{216}(72x \bmod 360) & = & 72x \bmod 5 \\ \theta_{225} : Z_8 & \to & \mathbb{Z}_8 & \theta_{225}(45x \bmod 360) & = & 45x \bmod 8 \\ \theta_{280} : Z_9 & \to & \mathbb{Z}_9 & \theta_{280}(40x \bmod 360) & = & 40x \bmod 9 \end{array}$$

$$\begin{array}{lcllcl} \theta_{216}^{-1} : \mathbb{Z}_5 & \to & Z_5 & \theta_{216}^{-1}(x \bmod 5) & = & 216x \bmod 360 \\ \theta_{225}^{-1} : \mathbb{Z}_8 & \to & Z_8 & \theta_{225}^{-1}(x \bmod 8) & = & 225x \bmod 360 \\ \theta_{280}^{-1} : \mathbb{Z}_9 & \to & Z_9 & \theta_{280}^{-1}(x \bmod 9) & = & 280x \bmod 360 \end{array}$$

The elements, e_5, e_8, and e_9 satisfy

$$e_5 \cdot e_8 \equiv e_5 \cdot e_9 \equiv e_8 \cdot e_9 \equiv 0 \bmod 360$$

and

$$e_5 + e_8 + e_9 \equiv 1 \bmod 360.$$

Therefore, for all $x \in \mathbb{Z}_{360}$, we have

$$xe_5 + xe_8 + xe_9 \equiv x \bmod 360$$

and this representation is unique. That is,

$$\mathbb{Z}_{360} = e_5\mathbb{Z}_{360} \oplus e_8\mathbb{Z}_{360} \oplus e_9\mathbb{Z}_{360}.$$

The other nonzero proper ideals of $\mathbb{Z}_{360}$ (besides Z_5, Z_8, Z_9) that are rings with addition and multiplication modulo 360 are

Ideal	*Identity*				
$Z_5 \oplus Z_8$	$e_5 + e_8$	$\equiv$	$216 + 225$	$\equiv$	$81 \bmod 360$
$Z_5 \oplus Z_9$	$e_5 + e_9$	$\equiv$	$216 + 280$	$\equiv$	$136 \bmod 360$
$Z_8 \oplus Z_9$	$e_8 + e_9$	$\equiv$	$225 + 280$	$\equiv$	$145 \bmod 360$

We have the following general result.

Corollary 11. *Let $\mathbb{Z}$ be the ring of integers, $p_1, p_2, \ldots, p_k$, be prime integers, $r_1, r_2, \ldots, r_k$, be positive integers. Set $b_i = p_i^{r_i}$ for $i = 1, 2, \ldots, k$, $a = \prod b_i$, and $a_i = \prod_{i \neq j} b_j$. Then,*

$$\mathbb{Z}_a = e_1\mathbb{Z}_a + e_2\mathbb{Z}_a + \cdots + e_{k-1}\mathbb{Z}_a + e_k\mathbb{Z}_a,$$

where e_i is the unique nonzero idempotent in the ideal $a_i\mathbb{Z}_a \subsetneq \mathbb{Z}_a$, e_i is orthogonal to e_j whenever $i \neq j$, and the sum is a direct sum.

Proof. Denote by $\mathfrak{a}$, $\mathfrak{a}_i$, and $\mathfrak{b}_i$ the ideals generated by a, a_i, and b_i respectively. The idempotent e_i is an element of $\mathfrak{a}_i$. The orthogonality of e_i and e_j, $i \neq j$, follows from Corollary 8. By Corollary 7, the ideal $\mathfrak{a}_i/\mathfrak{a}$ is isomorphic to $\mathbb{Z}/\mathfrak{b}_i = \mathbb{Z}_{b_i}$ which is a local ring. The uniqueness of e_i follows from Lemma 2. The direct sum decomposition of $\mathbb{Z}_a$ follows from Corollary 9. □

It follows from Lemma 5, that given positive integers a, and b such that $\gcd(a, b) = 1$ then the ring

$$E_a = b\mathbb{Z}_a = \{0, b, 2b, \ldots, ab - b\}$$

is a ring under addition and multiplication modulo ab that is isomorphic to $\mathbb{Z}_a$. The ring E_a has multiplicative identity $(b^{\phi(a)} \bmod ab)$. This construction can be done for an infinite number of integers b but since it could happen that $(b^{\phi(a)} \neq b \bmod ab)$, b may not the multiplicative identity of E_a. However, we have the following proposition.

Proposition 12. *Let e be an integer, $e \geq 3$. Let b be a divisor of e, and $a > 1$ be a divisor of $e - 1$. Then, e is a nonzero idempotent in the ring $\mathbb{Z}_{ab}$ and is the identity of the ring E_a. Moreover, the map*

$$E_a \to \mathbb{Z}_a,\ x \bmod ab \mapsto x \bmod a,$$

is an isomorphism.

Proof. The proposition follows from Lemma 5 since $\gcd(a, b) = 1$. □

Next we observe that, if an integer $e \geq 3$ is an idempotent modulo n then $e(e - 1) = 0 \bmod n$. This implies that n must be a divisor of $e(e - 1)$. Since we want e to be a nonzero idempotent modulo a, n must be of the form $n = ab$ with a and b chosen as in the proposition. This proves the following corollary.

Corollary 13. *Let e be an integer, $e \geq 3$, and N_1 and N_2 be the number of divisors of e and $e - 1$ respectively. Then there are $N_1(N_2 - 1)$ integers of the form ab so that $(e \bmod ab)$ is the identity of the ring E_a.*

Example 14. *Consider the integer $e = 2016$. Since $2016 = 2^5 \cdot 3^2 \cdot 7$ and $2015 = 5 \cdot 13 \cdot 31$ we have $N_1 = 36$ and $N_2 = 8$. Therefore, there are $36 \cdot 7 = 252$ integers of the form ab (ab as in Proposition 12) such that $(2016 \bmod ab)$ is the identity of the ring*

$$E_a = \{0, e, 2e, \ldots, ae - e\}.$$

1. *Let $a = 5$ and $b = 2$. Then*

$$\begin{aligned} E_5 &= \{0, 2016, 4032, 6048, 8064\} \\ &= \{0, 6, 2, 8, 4\} \\ &= \{0, 2, 4, 6, 8\} \end{aligned}$$

   endowed with addition and multiplication modulo $10 = 2 \cdot 5$ is a ring with multiplicative identity $6 = 2016 \bmod 10$. E_5 is isomorphic to $\mathbb{Z}_5$.

2. *Let $a = 65$ and $b = 6$. Then*

$$\begin{aligned} E_{65} &= \{0, 2016, 4032, \ldots, 127008, 129024\} \\ &= \{0, 66, 132, \ldots, 258, 324\} \\ &= \{0, 6, 12, \ldots, 378, 384\} \end{aligned}$$

   endowed with addition and multiplication modulo $390 = 6 \cdot 65$ is a ring with multiplicative identity $66 = 2016 \bmod 390$. E_{65} is isomorphic to $\mathbb{Z}_{65}$.

3. *Let $a = 31$ and $b = 2016$. Then,*

$$E_{31} = \{0, 2016, 4032, \ldots, 58464, 60480\}$$

   endowed with addition and multiplication modulo $62496 = 31 \cdot 2016$ is a ring with multiplicative identity 2016. E_{31} isomorphic to the field $\mathbb{Z}_{31}$.

4. *Let $a = 7$ and $b = 288$. Since $288 - 1 = 287 = 7 \cdot 41$, then*

$$E_7 = \{0, 288, 576, 864, 1152, 1440, 1728\}$$

   endowed with addition and multiplication modulo $2016 = 7 \cdot 288$ is a ring with multiplicative identity 288.

4 Factor rings of the ring of Gaussian integers

In this section we will recall some properties of the Gaussian integers, define and compute an Euler function φ_G and give some examples of factor rings of the ring of Gaussian integers. First, recall that the Gaussian integers (denoted by $\mathbb{Z}[\mathbf{i}]$) is the sub-ring of the field of complex numbers given below

$$\mathbb{Z}[\mathbf{i}] = \{x + y\mathbf{i} \,|\, x \text{ and } y \text{ integers, } \mathbf{i}^2 = -1\}$$

endowed with addition and multiplication inherited from the field of complex number. The ring $\mathbb{Z}[\mathbf{i}]$ is an Euclidean domain with Euclidean function

$$\lambda : \mathbb{Z}[\mathbf{i}] \to \{0, 1, 2, 3, \ldots\}, \;\; \lambda(x + y\mathbf{i}) = x^2 + y^2.$$

Since $\mathbb{Z}[\mathbf{i}]$ is an Euclidean domain we have a division algorithm on it.

Theorem 15 (Division Algorithm for Gaussian Integers)**.** *Let $z_1 \neq 0$ and z_2 be Gaussian integers then there exist q and r Gaussian integers such that*

$$z_2 = qz_1 + r, \text{and } r = 0 \text{ or } \lambda(r) < \lambda(z_1).$$

Proof. Let $d = \lambda(z_1) \in \mathbb{Z}$. Write $z_2\overline{z}_1 = A + B\mathbf{i}$. By the division algorithm on the ring of integers, we can write A and B as $A = q_1 d + r_1$ and $B = q_2 d + r_2$ with $-\frac{d}{2} \leq r_1 \leq \frac{d}{2}$ and $-\frac{d}{2} \leq r_2 \leq \frac{d}{2}$. Therefore,

$$z_2\overline{z}_1 = A + B\mathbf{i} = (q_1 + q_2)d + (r_1 + r_2\mathbf{i}) = (q_1 + q_2\mathbf{i})z_1\overline{z}_1 + (r_1 + r_2\mathbf{i}).$$

Since $\overline{z}_1$ divides $z_2\overline{z}_1$ and $(q_1 + q_2\mathbf{i})z_1\overline{z}_1$, it follows that $\overline{z}_1$ divides $r_1 + r_2\mathbf{i}$. That is,

$$r = \frac{r_1 + r_2\mathbf{i}}{\overline{z}_1}$$

is a Gaussian integer. This shows that

$$z_2 = (q_1 + q_2\mathbf{i})z_1 + r.$$

Since, $-\frac{d}{2} \leq r_1 \leq \frac{d}{2}$ and $-\frac{d}{2} \leq r_2 \leq \frac{d}{2}$ either $r = 0$ or

$$\lambda(r) = \lambda\left(\frac{r_1 + r_2\mathbf{i}}{\overline{z}_1}\right) = \frac{\lambda(r_1 + r_2\mathbf{i})}{\lambda(\overline{z}_1)} = \frac{r_1^2 + r_2^2}{d} \leq \frac{(d/2)^2 + (d/2)^2}{d} = \frac{d^2/2}{d} = \frac{d}{2} < \lambda(z_1). \quad \square$$

The ring $\mathbb{Z}[\mathbf{i}]$, being an Euclidean domain, is a unique factorization domain. Therefore, if a is a Gaussian integer then a can be written uniquely (up to units) as the product of prime Gaussian elements. The following theorem characterizes the prime elements of $\mathbb{Z}[\mathbf{i}]$.

Theorem 16. *Let p be a prime in $\mathbb{Z}$. Then:*

1. *If $p = 2$, then $1 + \mathbf{i}$ is prime in $\mathbb{Z}[\mathbf{i}]$ and $2 = \mathbf{i}^3(1 + \mathbf{i})^2$.*
2. *If $p \equiv 3 \bmod 4$, then p remains prime in $\mathbb{Z}[\mathbf{i}]$.*
3. *If $p \equiv 1 \bmod 4$, then there exists a prime $\pi \in \mathbb{Z}[\mathbf{i}]$ such that $p = \pi\overline{\pi}$, and the primes π and $\overline{\pi}$ are nonassociate in $\mathbb{Z}[\mathbf{i}]$. Furthermore, every prime in $\mathbb{Z}[\mathbf{i}]$ is associate to one of the primes listed in (1)-(3) above. (Two primes are associate if they differ by a unit factor.)*

Proof. See [3] page 81. $\square$

Denote by $\langle a \rangle$ the ideal of $\mathbb{Z}[\mathbf{i}]$ generated by a and by $\mathbb{Z}[\mathbf{i}]_a$ the quotient ring $\mathbb{Z}[\mathbf{i}]/\langle a \rangle$. We have the following result.

Corollary 17. *Let $\mathbb{Z}[\mathbf{i}]$ be the ring of Gaussian integers, $\pi_1, \pi_2, \ldots, \pi_k$ be prime elements in $\mathbb{Z}[\mathbf{i}]$, $r_1, r_2, \ldots, r_k$ be positive integers, $a_l = \pi_l^{r_l}$ for $l = 1, 2, \ldots, k$, $a = \prod a_l$, and $b_l = \prod_{l \neq j} a_j$. Then*

$$\mathbb{Z}[\mathbf{i}]_a = e_1\mathbb{Z}[\mathbf{i}]_{b_1} + e_2\mathbb{Z}[\mathbf{i}]_{b_2} + \cdots + e_{k-1}\mathbb{Z}[\mathbf{i}]_{b_{k-1}} + e_k\mathbb{Z}[\mathbf{i}]_{b_k}$$

where e_l is the unique nonzero idempotent in the ideal $b_l\mathbb{Z}[\mathbf{i}]_a \subsetneq \mathbb{Z}[i]_a$, e_l is orthogonal to e_j whenever $l \neq j$, and the sum is a direct sum.

Proof. The proof is similar to the proof of Corollary 11, so we omit it. $\square$

We remark that Proposition 12 and Corollary 13 remain valid in the Gaussian integers setting. We also have the following theorem about factor rings of the ring $\mathbb{Z}[\mathbf{i}]$.

Theorem 18. *Let $z = a + b\mathbf{i}$ be a Gaussian integer then*

1. $\mathbb{Z}[\mathbf{i}]_{a+b\mathbf{i}} \cong \mathbb{Z}[\mathbf{i}]_{-a-b\mathbf{i}} \cong \mathbb{Z}[\mathbf{i}]_{b-a\mathbf{i}} \cong \mathbb{Z}[\mathbf{i}]_{-b+a\mathbf{i}}$
2. *If $a > 1$ and $b = 0$ then $\mathbb{Z}[\mathbf{i}]_{a+b\mathbf{i}} = \mathbb{Z}[\mathbf{i}]_a \cong \mathbb{Z}_a[\mathbf{i}]$*
3. *If $\gcd(a, b) = 1$ then $\mathbb{Z}[\mathbf{i}]_{a+b\mathbf{i}} = \mathbb{Z}_{a^2+b^2}$*
4. *If $n > 0$ is an integer then*
   (a) *If $n = 2m$, $\mathbb{Z}[\mathbf{i}]_{(1+\mathbf{i})^n} \cong \mathbb{Z}_{2^m}[\mathbf{i}]$.*
   (b) *If $n = 2m + 1$, $m \geq 1$ then $\mathbb{Z}[\mathbf{i}]_{(1+\mathbf{i})^n} \cong \mathbb{Z}[x]/\langle 2^m x, 2^{m+1}, x^2 + x + 2\rangle$. In this case, $\mathbb{Z}[\mathbf{i}]_{(1+\mathbf{i})^n}$ is not isomorphic to $\mathbb{Z}_c$, $\mathbb{Z}_c[\mathbf{i}]$, or to any direct product of rings of this type.*

Proof. See [4] Fact 1 and Theorems 1, 2, and 5. □

Example 19. *Consider the ring $\mathbb{Z}[\mathbf{i}]_{360}$. Let $z_1 = 1 + 2\mathbf{i}$. Since $360 = \mathbf{i} \cdot z_1 \cdot \overline{z}_1 \cdot 3^2 \cdot (1 + \mathbf{i})^6$ The sets*

$$\begin{array}{rcccl} Z_5 &=& \{x72(1+2\mathbf{i}) \mid x \in \mathbb{Z}_5\} &=& 72(1+2\mathbf{i})\mathbb{Z}[\mathbf{i}]_{360}, \\ \overline{Z}_5 &=& \{x72(1-2\mathbf{i}) \mid x \in \mathbb{Z}_5\} &=& 72(1-2\mathbf{i})\mathbb{Z}[\mathbf{i}]_{360}, \\ Z_8[\mathbf{i}] &=& \{45(x+y\mathbf{i}) \mid x, y \in \mathbb{Z}_8\} &=& 45\mathbb{Z}[\mathbf{i}]_{360}, \text{ and} \\ Z_9[\mathbf{i}] &=& \{40(x+y\mathbf{i}) \mid x, y \in \mathbb{Z}_9\} &=& 40\mathbb{Z}[\mathbf{i}]_{360} \end{array}$$

are principal ideals of $\mathbb{Z}[\mathbf{i}]_{360}$. These ideals are rings with unity under addition and multiplication modulo 360. The multiplicative identities of the ideals Z_5, $\overline{Z}_5$, Z_8 and Z_9 are $e_5 = 288 - 144\mathbf{i}$, $\overline{e}_5 = 288 + 144\mathbf{i}$, $e_8 = 225$, and $e_9 = 280$ respectively. The element 0 is the additive identity for them. The ring isomorphisms and their inverses are given below.

$$\begin{array}{rclrcl} \theta_{e_5} : Z_5 &\to& \mathbb{Z}[\mathbf{i}]_{1-2\mathbf{i}} & \theta_{e_5}(x72(1+2\mathbf{i}) \bmod 360) &=& x72(1+2\mathbf{i}) \bmod 1-2\mathbf{i} \\ \theta_{\overline{e}_5} : \overline{Z}_5 &\to& \mathbb{Z}[\mathbf{i}]_{1+2\mathbf{i}} & \theta_{\overline{e}_5}(x72(1-2\mathbf{i}) \bmod 360) &=& x72(1-2\mathbf{i}) \bmod 1+2\mathbf{i} \\ \theta_{225} : Z_8[\mathbf{i}] &\to& \mathbb{Z}_8[\mathbf{i}] & \theta_{225}(45(x+y\mathbf{i}) \bmod 360) &=& 45(x+y\mathbf{i}) \bmod 8 \\ \theta_{280} : Z_9[\mathbf{i}] &\to& \mathbb{Z}_9[\mathbf{i}] & \theta_{280}(40(x+y\mathbf{i}) \bmod 360) &=& 40(x+y\mathbf{i}) \bmod 9 \end{array}$$

$$\begin{array}{rclrcl} \theta_{e_5}^{-1} : \mathbb{Z}[\mathbf{i}]_{1-2\mathbf{i}} &\to& Z_5 & \theta_{e_5}^{-1}(x \bmod 1-2\mathbf{i}) &=& (288-144\mathbf{i})x \bmod 360 \\ \theta_{\overline{e}_5}^{-1} : \mathbb{Z}[\mathbf{i}]_{1+2\mathbf{i}} &\to& \overline{Z}_5 & \theta_{\overline{e}_5}^{-1}(x \bmod 1-2\mathbf{i}) &=& (288+144\mathbf{i})x \bmod 360 \\ \theta_{225}^{-1} : \mathbb{Z}_8[\mathbf{i}] &\to& Z_8[\mathbf{i}] & \theta_{225}^{-1}(x+y\mathbf{i}) &=& 225(x+y\mathbf{i}) \bmod 360 \\ \theta_{280}^{-1} : \mathbb{Z}_9[\mathbf{i}] &\to& Z_9[\mathbf{i}] & \theta_{280}^{-1}(x+y\mathbf{i}) &=& 280(x+y\mathbf{i}) \bmod 360 \end{array}$$

The elements, e_5, $\overline{e}_5$, e_8, and e_9 satisfy

$$e_5 \cdot \overline{e}_5 \equiv e_5 \cdot e_8 \equiv e_5 \cdot e_9 \equiv \overline{e}_5 \cdot e_8 \equiv \overline{e}_5 \cdot e_9 \equiv e_8 \cdot e_9 \equiv 0 \bmod 360$$

and

$$e_5 + \overline{e}_5 + e_8 + e_9 \equiv 1 \bmod 360$$

Therefore, for all $x \in \mathbb{Z}[\mathbf{i}]_{360}$, we have

$$xe_5 + x\overline{e}_5 + xe_8 + xe_9 \equiv x \bmod 360$$

and this representation is unique. That is,

$$\mathbb{Z}[\mathbf{i}]_{360} = e_5\mathbb{Z}[\mathbf{i}]_{360} \oplus \overline{e}_5\mathbb{Z}[\mathbf{i}]_{360} \oplus e_8\mathbb{Z}_{360} \oplus e_9\mathbb{Z}_{360}.$$

The other nonzero proper ideals of $\mathbb{Z}_{360}$ (besides Z_5, $\overline{Z}_5$, Z_8, Z_9) that are rings with addition and multiplication modulo 360 are

Ideal	*Identity*				
$Z_5 \oplus \overline{Z}_5$	$e_5 \oplus \overline{e}_5$	$\equiv$	$288 - 144\mathbf{i} + 288 + 144\mathbf{i}$	$\equiv$	$216 \bmod 360$
$Z_5 \oplus Z_8$	$e_5 \oplus e_8$	$\equiv$	$288 - 144\mathbf{i} + 225$	$\equiv$	$153 - 144\mathbf{i} \bmod 360$
$Z_5 \oplus Z_9$	$e_5 \oplus e_9$	$\equiv$	$288 - 144\mathbf{i} + 280$	$\equiv$	$218 - 144\mathbf{i} \bmod 360$
$\overline{Z}_5 \oplus Z_8$	$\overline{e}_5 \oplus e_8$	$\equiv$	$288 + 144\mathbf{i} + 225$	$\equiv$	$153 + 144\mathbf{i} \bmod 360$
$\overline{Z}_5 \oplus Z_9$	$\overline{e}_5 \oplus e_9$	$\equiv$	$288 + 144\mathbf{i} + 280$	$\equiv$	$218 + 144\mathbf{i} \bmod 360$
$Z_8 \oplus Z_9$	$e_8 \oplus e_9$	$\equiv$	$225 + 280$	$\equiv$	$145 \bmod 360$
$Z_5 \oplus \overline{Z}_5 \oplus Z_8$	$e_5 \oplus \overline{e}_5 \oplus e_8$	$\equiv$	$288 - 144\mathbf{i} + 288 + 144\mathbf{i} + 225$	$\equiv$	$81 \bmod 360$
$Z_5 \oplus \overline{Z}_5 \oplus Z_9$	$e_5 \oplus \overline{e}_5 \oplus e_9$	$\equiv$	$288 - 144\mathbf{i} + 288 + 144\mathbf{i} + 280$	$\equiv$	$136 \bmod 360$
$Z_5 \oplus Z_8 \oplus Z_9$	$e_5 \oplus e_8 \oplus e_9$	$\equiv$	$288 - 144\mathbf{i} + 225 + 280$	$\equiv$	$73 - 144\mathbf{i} \bmod 360$
$\overline{Z}_5 \oplus Z_8 \oplus Z_9$	$\overline{e}_5 \oplus e_8 \oplus e_9$	$\equiv$	$288 + 144\mathbf{i} + 225 + 280$	$\equiv$	$73 + 144\mathbf{i} \bmod 360$

Example 20. *Consider the integer $e = 2017$. Since we have the following prime power decomposition of 2017 and 2016 over the Gaussian integers*

$$2017 = (9 + 44\mathbf{i})(9 - 44\mathbf{i}) \text{ and } 2016 = 2^5 \cdot 3^2 \cdot 7 = \mathbf{i}^3(1 + \mathbf{i})^{10} \cdot 3^2 \cdot 7$$

we have $N_1 = 4$ and $N_2 = 66$. Therefore, there are $4 \cdot 66 = 264$ Gaussian integers of the form ab, where a is a divisor of 2017 and b is a divisor of 2016, so that $(e \bmod ab)$ is the identity of the ring $E_a = b\mathbb{Z}[\mathbf{i}]_a$ endowed with addition and multiplication modulo ab.

5 Factor rings of rings of polynomials

Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ be the ring of polynomials with coefficients in $\mathbb{F}$ and indeterminate x. The ring $\mathbb{F}[x]$ is an Euclidean domain with Euclidean function

$$\deg : \mathbb{F}[x] \to \mathbb{Z}^+ \cup \{0\}$$

where $\deg(f(x))$ is the degree of the polynomial $f(x)$.

Theorem 21 (Division Algorithm for Polynomial Rings)**.** *Let $d(x) \neq 0$ and $f(x)$ be elements of $\mathbb{F}[x]$ then there exists $q(x)$ and $r(x)$ elements of $\mathbb{F}[x]$ such that*

$$f(x) = q(x)d(x) + r(x), \text{and } r(x) = 0 \text{ or } \deg(r(x)) < \deg(d(x)).$$

Moreover, $q(x)$ and $r(x)$ are unique.

Proof. See [5] Theorem 16.2. □

Since $\mathbb{F}[x]$ is an Euclidean domain, it is a unique factorization domain. We have the following result.

Corollary 22. *Let $p_1(x), p_2(x), \dots, p_k(x)$, be irreducible polynomials in $\mathbb{F}[x]$, $r_1, r_2, \dots, r_k$, be positive integers. Set $b_i(x) = p_i(x)^{r_i}$ for $i = 1, 2, \dots, k$, $a(x) = \prod b_i(x)$, and $a_i(x) = \prod_{i \neq j} b_j(x)$. Let $\mathbb{F}[x]_{a(x)} := \mathbb{F}[x]/\langle a(x)\rangle$. Then,*

$$\mathbb{F}[x]_{a(x)} = e_1(x)\mathbb{F}[x]_{a(x)} + e_2(x)\mathbb{F}[x]_{a(x)} + \cdots + e_{k-1}(x)\mathbb{F}[x]_{a(x)} + e_k(x)\mathbb{F}[x]_{a(x)},$$

where $e_i(x)$ is the unique nonzero idempotent in the ideal $a_i(x)\mathbb{F}[x]_{a(x)} \subsetneq \mathbb{F}[x]_{a(x)}$, $e_i(x)$ is orthogonal to $e_j(x)$ whenever $i \neq j$, and the sum is a direct sum.

Proof. The proof is similar to the proof of Corollary 11. □

Example 23. *Let p be a prime and $q = p^r$. Let $\mathbb{F}_q$ be the finite field with q elements. Let n be a positive integer with $\gcd(p, n) = 1$ and $a(x) = x^n - 1$ be an element of $\mathbb{F}_q[x]$. The condition $\gcd(p, n) = 1$ implies that the factorization of $a(x)$ has not repeated factors, that is,*

$$a(x) = x^n - 1 = p_1(x) \cdot p_2(x) \cdots p_{k-1}(x) \cdot p_k(x)$$

where $p_i(x)$ is irreducible of degree d_i and $p_i(x) \neq p_j(x)$ if $i \neq j$. Corollary 22 implies that the factor ring $\mathbb{F}_q[x]_{a(x)} := \mathbb{F}_q[x]/\langle a(x)\rangle$ decomposes as the direct sum

$$\mathbb{F}_q[x]_{a(x)} = e_1(x)\mathbb{F}_q[x]_{a(x)} + e_2(x)\mathbb{F}_q[x]_{a(x)} + \cdots + e_{k-1}(x)\mathbb{F}_q[x]_{a(x)} + e_k(x)\mathbb{F}_q[x]_{a(x)}.$$

The ideal $e_j(x)\mathbb{F}_q[x]_{a(x)}$ is a field under addition and multiplication modulo $a(x)$, it is isomorphic to the field with q^{d_i} elements $\mathbb{F}_q[x]_{p_i(x)}$, and the idempotent element $e_j(x)$ is given by the formula $e_j(x) = a_j(x)^{q^{d_i}-1} \bmod a(x)$. The latter statement follows from the fact that the order of the group of units $U\left(\mathbb{F}_q[x]_{p_i(x)}\right)$ of the field $\mathbb{F}_q[x]_{p_i(x)}$ is $q^{d_i} - 1$. The idempotent elements $e_j(x)$ are known as primitive idempotent elements and they generate the minimal ideals of the factor ring $\mathbb{F}_q[x]_{a(x)}$.

This example is important because the ideals of the ring $\mathbb{F}_q[x]_{a(x)}$ correspond to $q-$ary cyclic error-correcting codes and the minimal idempotent elements correspond to the $q-$ary minimal cyclic codes. See Theorem 8.1 in [1] for a proof of this correspondence.

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